Asymptotic analysis of classical wave localization in multiple-scattering random media

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In this work we consider the localization of classical waves propagating in random continuum. We apply the method of proper time for the transfer from the elliptic-type wave equation to the generalized parabolic one. Presenting the solution of the latter equation in the form of the Feynman path integral allows us to estimate the so-called wave correction terms. These corrections are related to coherent backscattering and recurrent multiple-scattering events, i.e., to phenomena that cannot be described within the framework of the conventional theories of radiative transfer or small-angle scattering. We evaluate the wave correction to the mean intensity of a point source located in a statistically homogeneous Gaussian random medium. Our results confirm that there is an essential difference between two- and three-dimensional systems. We consider both isotropic and anisotropic media and show in particular that in the latter case there is a critical value of the anisotropy parameter, below which the system behaves basically as a three-dimensional isotropic medium, i.e., the wave correction is positive for all observation angles. Above this critical value the properties of the medium are similar to those of a layered structure. [S1063-651X(97)04111-1]

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I. INTRODUCTION

Long-term investigations of classical wave propagation in random media have exposed principally two main approaches to the effective analysis of a wide spectrum of associated phenomena [1–3]. The first of these general approaches, the radiative transfer theory, is based on the phenomenological foundation according to which the total effect of multiple fields involved in some elementary volume can be found by a simple noncoherent summation of their energy fluxes. In this case the transport of the field has a diffusive character and is governed by intuition-grounded laws related to irreversible processes. Another approach is based on the exact wave formulation and accounts, in principle, for all the wave-nature effects neglected in the transport theory.

For a long time it has been assumed that the reduction of the wave picture to the radiation transport under the condition of multiple scattering is straightforward. However, in recent decades the use of the wave formulation for the statistical assertion of the transport theory has led to some difficulties in describing the wave scattering in the backward direction [4–8]. In particular, Watson discovered that for any configuration of scattering centers there exist direct and reversed scattering paths, for which coherent interference of the waves is possible [5]. This constructive interference leads to the enhancement of backscattered radiation as compared to the result predicted in the framework of the radiative transfer theory. In general, it has been understood that coherence effects are important even after many scattering events. Further investigations have led to the formulation of the concept of the so-called wave correction terms [7], which are responsible for some nontrivial effects related to the time symmetry of the wave equations and the consequent reversibility of the wave processes.

At the same time, after the discovery of the possibility of electron localization in disordered solids (strong or Anderson-type localization) [9], it was recognized that the main results obtained in quantum mechanics could be transferred to classical (acoustic or electromagnetic) waves [10–12] (for a review see Ref. [13]). Strong localization takes place if the elastic scattering length l_s is of the order of the wavelength λ, such that the characteristic width of the backscattering peak, which is equal to about λ/l_s, enlarges to the back hemisphere. In this case the wave does not diffuse to infinity but is trapped within a bounded spatial region near the source. For this reason the backscattering enhancement of classical waves, which was experimentally observed in discrete random media [14–16], was reported originally as weak localization, the precursor of the strong (Anderson-type) one.

Several methods based on looking at the transport properties of a multiply scattered wave have been developed for the theoretical description of the localization phenomenon in disordered systems [10,11,17–21]. In these works the localization is concerned with the vanishing of the diffusion constant. Therefore, a natural way to search for the localization regime is to study the long-time behavior of the average two-particle Green's function far from the source, which corresponds to the low-frequency low-wave-number limit of the diffusion constant. As a result of numerous investigations it was established in particular that the dimension of the disordered medium is a crucial parameter. In one and two dimensions any degree of disorder leads to a finite localization length, while in three dimensions a certain critical degree of disorder is needed before localization will be observed [13].

In order to consider the localization behavior of a classical wave, we apply here another method analyzing a stationary problem, namely, studying asymptotically the mean intensity distribution in the far field of a point source located in an unbounded statistically homogeneous random medium.

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For the analysis we use the following procedure. First, we apply the method of proper time originally proposed for the integration of quantum-mechanical equations by Fock [22] and later developed by Schwinger [23] (see also Ref. [24]). The method is based on the introduction of an additional pseudotime variable and a transfer to a higher-dimensional space, in which the propagation process is described by the generalized parabolic equation similar to the nonstationary Schrödinger equation in quantum mechanics. We present its solution in the form of a Feynman path integral [25], the asymptotic evaluation of which in the far field allows us, in principle, to estimate the wave corrections for all statistical moments of the field. These corrections are related to coherent backscattering and, using the terminology of [26], also to recurrent multiple-scattering events, i.e., to phenomena that cannot be described in the framework of the conventional radiative transfer theory. For a three-dimensional isotropic medium this program has been realized in our recent paper [27]. In particular, evaluating the second statistical moment of the field (average two-particle Green's function), we have obtained that the normalized mean intensity differs from unity. We have related such a behavior to the localization phenomenon. In the present work we further develop these results by studying wave correction in both two- and three-dimensional (2D and 3D) random media, characterized by isotropic and anisotropic correlation functions.

The outline of this work is as follows. First, in Sec. II we introduce the generalized parabolic equation and present its solution in a path-integral form. In order to reduce the functional integral to a finite N-dimensional one we parametrize the trajectory, expanding each virtual path into an eigenfunction series. Next, using a perturbative technique and representing the unknown function as a sum of a leading term plus a correction, we obtain an asymptotic expression for the mean intensity of the wave in the far field. This result, which is valid for arbitrary dimension of the medium, is presented in Sec. III. A particular case of one-dimensional system is briefly discussed. Further, in Sec. IV we analyze the wave correction for isotropic media in both two and three dimensions. To exemplify the results we evaluate the correction for a Gaussian correlation function with a characteristic scale l_ε. In three dimensions the wave correction is positive and has a quite narrow window in which the wavelength is comparable to the correlation scale l_ε. It is the intermediate spectral window that possibly separates extended states at both higher and lower frequencies within the framework of the wave localization concept. In two dimensions the correction is mainly negative, which can serve as indication of the Anderson localization. Section V is devoted to the analysis of the localization in anisotropic media, which are the intermediate case between isotropic media and purely layered ones. The results obtained there seem to be the more important part of this work. We show that the wave correction in a 2D anisotropic medium does not depend on the observation angle. For the 3D problem the directions along and across the quasilayered structure are essentially distinguished and the localization of the wave energy along the layers is observed. The dependence of the wave correction on the anisotropy parameter μ manifests a critical behavior. The results show that there is a critical value of μ, below which the medium behaves basically as a three-dimensional isotropic medium, i.e., the wave correction is positive for all observation angles. Above this critical value the properties of the medium are similar to those of a layered structure. Finally, Sec. VI contains a summary and some principal concluding remarks.

II. PATH-INTEGRAL REPRESENTATION

We start with the reduced Helmholtz equation describing the propagation and scattering of scalar time-harmonic waves in an inhomogeneous medium. The Green's function is defined by

$$\nabla^2 G(\mathbf{R}|\mathbf{R}_0) + k^2[1+\tilde{\varepsilon}(\mathbf{R})]G(\mathbf{R}|\mathbf{R}_0) = -\delta(\mathbf{R}-\mathbf{R}_0), \tag{2.1}$$

where $\mathbf{R}$ denotes the position vector in m-dimensional space ($m=2$ or 3), k is the wave number associated with the homogeneous medium, and $\varepsilon(\mathbf{R}) = 1+\tilde{\varepsilon}(\mathbf{R})$ is the random permittivity distribution. We suppose that while ε is a real function, k contains an infinitesimally small positive imaginary part that provides the convergence of some integrals appearing in the course of the work. Equation (2.1) is known to serve as a reasonable model for acoustic wave propagation and also for some electromagnetic problems in which the polarization effects can be neglected.

Let us introduce an auxiliary parabolic equation

$$2ik\partial_\tau g + \nabla^2 g + k^2\tilde{\varepsilon}(\mathbf{R})g(\mathbf{R},\tau|\mathbf{R}_0,\tau_0) = 0, \quad \tau > \tau_0, \tag{2.2a}$$

with the initial condition

$$g(\mathbf{R},\tau_0|\mathbf{R}_0,\tau_0) = \delta(\mathbf{R}-\mathbf{R}_0). \tag{2.2b}$$

Then the Green's function $G(\mathbf{R}|\mathbf{R}_0)$ can be defined through the solution of Eq. (2.2) as

$$G(\mathbf{R}|\mathbf{R}_0) = \frac{i}{2k}\int_{\tau_0}^{\infty} d\tau \, \exp\left[i\,\frac{k}{2}\,(\tau-\tau_0)\right] g(\mathbf{R},\tau|\mathbf{R}_0,\tau_0). \tag{2.3}$$

The generalized parabolic equation (2.2) for the Green's function $g(\mathbf{R},\tau|\mathbf{R}_0,\tau_0)$ coincides with the nonstationary Schrödinger equation in quantum mechanics. Using this analogy, the solution of Eq. (2.2) can be presented in the Feynman path-integral form [25]

$$g(\mathbf{R},\tau|\mathbf{R}_0,\tau_0) = \int_{\mathbf{R}(\tau_0)=\mathbf{R}_0}^{\mathbf{R}(\tau)=\mathbf{R}} D\mathbf{R}(t)\exp\{iS[\mathbf{R}(t)]\}, \tag{2.4}$$

where the integration $\int D\mathbf{R}(t)$ in the continuum of possible trajectories is interpreted as a sum of contributions of arbitrary paths over which a wave propagates from the point $\mathbf{R}_0$ at the moment τ_0 to the point $\mathbf{R}$ at the moment τ and the functional

$$S[\mathbf{R}(t)] = \frac{k}{2}\int_{\tau_0}^{\tau} dt\{[\dot{\mathbf{R}}(t)]^2 + \tilde{\varepsilon}\,[\mathbf{R}(t)]\} \tag{2.5}$$

can be related to the phase accumulated along the corresponding path.

The path integral can be exactly handled only for Gaussian integrands, i.e., for the action $S[\mathbf{R}(t)]$ of a quadratic type [28]. It is clear that investigation of disordered media requires an approximate evaluation of the path integral. The commonly used strategy suggests reducing the original integral to a Gaussian form, which can be accomplished by a perturbative technique [28]. In general, we follow this way. Specifically, using the spectral form of the scattering potential and keeping only the first terms in the series expansion of the exponential, we present the double path-integral related to the mean intensity of the wave in a soluble Gaussian form (see Sec. III). The next step, the evaluation of this integral, can be performed by a variety of methods [25,28,29], all of them leading, of course, to the same results. We have deliberately chosen, nevertheless, a specific approach, known as orthogonal path expansion, because, in addition to the final results, it provides useful information about the relative weight of arbitrary scattering processes contributing to the unknown wave correction. According to this method we first extract from the virtual path

$$\mathbf{R}(t)=\overline{\mathbf{R}}(t)+\mathbf{Q}(t) \tag{2.6}$$

the classical trajectory

$$\overline{\mathbf{R}}(t)=\frac{\tau-t}{\tau-\tau_0}\,\mathbf{R}_0+\frac{t-\tau_0}{\tau-\tau_0}\,\mathbf{R}, \tag{2.7}$$

which is simply the straight line connecting the points $\mathbf{R}_0$ and $\mathbf{R}$, and expand each curved path $\mathbf{Q}(t)$ into the series

$$\mathbf{Q}(t)=\sum_{n=1}^{\infty}\psi_n(t)\mathbf{Q}_n, \tag{2.8}$$

where $\psi_n(t)$ is a complete set of orthogonal functions

$$\psi_n(t)=\frac{\sqrt{2(\tau-\tau_0)}}{\pi n}\sin\left(\frac{\pi n t}{\tau-\tau_0}\right). \tag{2.9}$$

Next we transfer from the path integral to integration over the coefficients $\mathbf{Q}_n$ of the orthogonal expansion. As a result, the path integral can be presented in the form

$$g(\mathbf{R},\tau|\mathbf{R}_0,\tau_0)=g_0(\mathbf{R},\tau|\mathbf{R}_0,\tau_0)g_\varepsilon(\mathbf{R},\tau|\mathbf{R}_0,\tau_0), \tag{2.10}$$

where g_0 is the free-space ($\tilde{\varepsilon}=0$) Green's function

$$g_0(\mathbf{R},\tau|\mathbf{R}_0,\tau_0)=\left[\frac{k}{2\pi i(\tau-\tau_0)}\right]^{m/2}\exp\left[\frac{ik(\mathbf{R}-\mathbf{R}_0)^2}{2(\tau-\tau_0)}\right] \tag{2.11}$$

and the inhomogeneous factor g_ε is a limit $N\to\infty$ of the finite-dimensional approximation

$$g_\varepsilon(\mathbf{R},\tau|\mathbf{R}_0,\tau_0)=\left(\frac{k}{2\pi i}\right)^{mN/2}\int d^{mN}Q_n\exp\left(i\,\frac{k}{2}\sum_{n=1}^{N}Q_n^2\right)\times\exp\left\{i\,\frac{k}{2}\int_{\tau_0}^{\tau}dt\;\tilde{\varepsilon}\left[\overline{\mathbf{R}}(t)+\sum_{n=1}^{N}\psi_n(t)\mathbf{Q}_n\right]\right\} \tag{2.12}$$

(here we use the notation $\underline{N}=1,2,\ldots,N$), or in a more compact form

$$g_\varepsilon(\mathbf{R},\tau|\mathbf{R}_0,\tau_0)=\oint D\mathbf{Q}(t)\exp\left\{i\,\frac{k}{2}\int_{\tau_0}^{\tau}dt\;\tilde{\varepsilon}\,[\overline{\mathbf{R}}(t)+\mathbf{Q}(t)]\right\}, \tag{2.13}$$

in which the circular integral is used to underline the fact that all the trajectories $\mathbf{Q}(t)$ are closed.

The generalized parabolic equation (2.2) satisfies the causality condition. This means that it describes the scattering process in the forward direction only, accounting for the trajectories, which do not have any turning point with respect to the auxiliary pseudotime coordinate τ. However, if we consider the projection of a consequent path onto the real m-dimensional space, we find that this formulation allows trajectories with multiple [Nth order in the finite-dimensional version (2.12) of the path integral] turning points, i.e., it takes into account all the backscattering and recurrent multiple-scattering events.

III. EVALUATION OF THE WAVE CORRECTION

Now we define the mean intensity of the wave at a point $\mathbf{R}$ as

$$\langle I(\mathbf{R}|\mathbf{R}_0)\rangle=\langle G(\mathbf{R}|\mathbf{R}_0\rangle G^*(\mathbf{R}|\mathbf{R}_0)\rangle, \tag{3.1}$$

where the angular brackets denote ensemble averaging. In a homogeneous medium the intensity distribution in the far field $k|\mathbf{R}-\mathbf{R}_0|\gg1$ is given approximately by the relation

$$I_0(\mathbf{R}|\mathbf{R}_0)=\tfrac{1}{4}k^{m-3}(2\pi|\mathbf{R}-\mathbf{R}_0|)^{1-m}, \tag{3.2}$$

which is exact for $m=3$. We restrict ourselves by considering a statistically homogeneous random medium, where the mean intensity is a function of the vector $\mathbf{L}=\mathbf{R}-\mathbf{R}_0$ only. It is clear that, within the framework of radiative transfer theory, the normalized mean intensity

$$\iota(\mathbf{L})=\langle I(\mathbf{L})\rangle/I_0(\mathbf{L}) \tag{3.3}$$

must be equal to unity, at least for statistically isotropic media. In order to estimate the constructive interference effects we start with integral representation (2.3), in which the Green's function $g(\mathbf{R},\tau|\mathbf{R}_0,\tau_0)$ is given by Eqs. (2.10)–(2.13). Next we approximate the normalized mean intensity in the far field by the relation

$$\iota(\mathbf{L})=\langle g_\varepsilon(\mathbf{R},\tau_0+L|\mathbf{R}_0,\tau_0)g_\varepsilon^*(\mathbf{R},\tau_0+L|\mathbf{R}_0,\tau_0)\rangle. \tag{3.4}$$

As shown [27], the error due to this approximation is essentially smaller than the correction that we try to evaluate. We assume that the random perturbations $\tilde{\varepsilon}(\mathbf{R})$ are Gaussian, which simplifies the averaging procedure. Then using the representation (2.13) and introducing the sum and difference vectors

$$\mathbf{P}(t)=\tfrac{1}{2}[\mathbf{Q}_1(t)+\mathbf{Q}_2(t)],\quad \mathbf{Q}(t)=\mathbf{Q}_1(t)-\mathbf{Q}_2(t), \tag{3.5}$$

we arrive at the expression

$$\iota(\mathbf{L})=\oint D\mathbf{P}(t)\oint D\mathbf{Q}(t) \times\exp\left[-\frac{k^2}{4}\int_0^L dt_1\int_0^L dt_2 F(t_1,t_2;\mathbf{P}(t),\mathbf{Q}(t))\right], \tag{3.6}$$

where the scattering function $F(t_1,t_2;\mathbf{P}(t),\mathbf{Q}(t))$ is given by

$$F(t_1,t_2;\mathbf{P}(t),\mathbf{Q}(t))=D_\varepsilon[\mathbf{R}_1(t_1)-\mathbf{R}_2(t_2)] -\tfrac{1}{2}\sum_{j=1}^{2} D_\varepsilon[\mathbf{R}_j(t_1)-\mathbf{R}_j(t_2)]. \tag{3.7}$$

In formula (3.7)

$$D_\varepsilon(\mathbf{R}_1-\mathbf{R}_2)=\langle[\tilde{\varepsilon}(\mathbf{R}_1)-\tilde{\varepsilon}(\mathbf{R}_2)]^2\rangle \tag{3.8}$$

is the structure function, which is related to the usually used correlation function

$$B_\varepsilon(\mathbf{R}_1-\mathbf{R}_2)=\langle\tilde{\varepsilon}(\mathbf{R}_1)\tilde{\varepsilon}(\mathbf{R}_2)\rangle \tag{3.9}$$

as

$$D_\varepsilon(\mathbf{R})=2[B_\varepsilon(\mathbf{0})-B_\varepsilon(\mathbf{R})] \tag{3.10}$$

if, obviously, $B_\varepsilon(\mathbf{0})$ exists. The vectors $\mathbf{R}_j(t)$ in Eq. (3.7) are defined as

$$\mathbf{R}_j(t)=\overline{\mathbf{R}}(t)+[\mathbf{P}(t)+(-1)^{j-1}\mathbf{Q}(t)/2]. \tag{3.11}$$

The projection of the trajectory $\mathbf{R}_j(t)$ onto the real space is the realization of a Brownian motion. Let us suppose for simplicity that the scattering potential $\tilde{\varepsilon}(\mathbf{R})$ is of a short-range type (δ correlated in m-dimensional space). If there are no self-(mutual) intersections of these projections, then the scattering function F is zero and $\iota(\mathbf{L})=1$, which is natural for the noncoherent transport picture. The condition of absence of intersections resembles the self-avoidance constraint on the paths in polymer physics [30]. In the case of classical waves this means that all scattering events of the propagation process are fully independent, which could take place only if the wave ''remembers''its way in order to avoid multiple visits to the same spatial points. Meanwhile, it is clear that the longer the trajectory, the higher the probability of self-intersection, and the fraction of self-avoiding paths is exponentially small for long trajectories [31]. Just these trajectories define the constructive interference effects and contribute to the nonzero value of the scattering function F.

Another consequence that can be predicted already is the essential role of dimensionality in possible manifestations of the constructive interference. In fact, as known from the theory of the self-avoiding walk, for $m\geqslant 4$ the self-avoiding constraint is relevant for a set of configurations of zero probability measure, so the Brownian path has no chance to intersect itself. In three dimensions we can anticipate only the double points, i.e., the points that are visited twice by the Brownian trajectory, while the multiple points of all orders are relevant in two dimensions [30].

In order to evaluate the influence of the path intersections we employ the following perturbative analysis. We assume that the contribution from the self-intersection points is rather small, which allows us to expand the exponential in Eq. (3.6) into a series and to keep the first two terms only. In this case the normalized mean intensity $\iota(\mathbf{L})$ can be presented as a sum of a leading term, which is simply unity, and some correction term χ, namely,

$$\iota(\mathbf{L})=1+\chi+\cdots . \tag{3.12}$$

Obviously, such a behavior can be recognized as a manifestation of the phenomenon of classical wave localization [13] and the investigation of this asymptote can help us in the description of the transition from extended to localized states. The wave correction is given as

$$\chi=\frac{k^2}{4}\int_0^L dt_1\int_0^L dt_2\oint D\mathbf{P}(t)\oint D\mathbf{Q}(t) \times F(t_1,t_2;\mathbf{P}(t),\mathbf{Q}(t)). \tag{3.13}$$

In order to obtain the soluble quadratic Lagrangian in the path integral we replace the structure function by its Fourier transform

$$D_\varepsilon(\mathbf{R})=2\int d^m\mathbf{K}[1-\exp(i\mathbf{R}\cdot\mathbf{K})]\Phi_\varepsilon(\mathbf{K}). \tag{3.14}$$

Then the path integral can be exactly evaluated and the correction takes the simple form

$$\chi=\frac{k^2}{2}\int_0^L dt_1\int_0^L dt_2\int d^mK\ \Phi_\varepsilon(\mathbf{K})\cos(\mathbf{T}\cdot\mathbf{K}) \times[\cos(\tilde{\eta}K^2)-\cos(\eta K^2)], \tag{3.15}$$

where $\mathbf{T}$ is the vector having the absolute value equal to $t=t_1-t_2$ and directed along the line connecting the source with the observation point. The functions $\eta(N)$ and $\tilde{\eta}(N)$ are presented by the series

$$\eta(N)=\frac{1}{2k}\sum_{n=1}^{N}[\psi_n(t_1)-\psi_n(t_2)]^2 \tag{3.16a}$$

and

$$\tilde{\eta}(N)=\frac{1}{2k}\sum_{n=1}^{N}[\psi_n^2(t_1)-\psi_n^2(t_2)]. \tag{3.16b}$$

Exact summation for $N\to\infty$ leads to

$$\eta=\frac{L}{2k}\frac{t}{L}\left(1-\frac{t}{L}\right) \tag{3.17a}$$

and

$$\tilde{\eta}=\frac{L}{2k}\frac{t}{L}\left(1-\frac{2t'}{L}\right), \tag{3.17b}$$

where $t'=\frac{1}{2}(t_1+t_2)$. The method of orthogonal path expansion allows us to estimate the relative role of arbitrary paths in their contribution to the wave correction. In general, the weight of each nth term in the series (3.16) decreases as $1/n^2$ and therefore the higher-order terms have to be negligible. However, a similar procedure, applied to a small-angle scattering in random media δ correlated along the propagation direction, has revealed a slow convergence to the exact result, with the error decreasing as $\sim N^{-\alpha}$, $\alpha<1$ [32]. Nevertheless, in that case even the first term alone has led to the qualitative description of the expected effects. This is not so in our studies when the multiplicity of the scattering process becomes the most important element of the wave localization phenomenon. In fact, the functions $\eta(N)$ and $\tilde{\eta}(N)$ become close to their exact versions given by Eqs. (3.17) only when a large number of eigenfunctions are taken into account. Physically this corresponds to a wave process that includes many backscattering and recurrent scattering events. Such a distinction between the exact version and its finite-term approximation can provide not only a quantitative difference in the solution but also dramatic changes in its behavior, including the sign of the final result.

Using now the exact expressions (3.17) and integrating in Eq. (3.15) over the difference coordinate t, we obtain

$$\chi=k^2\int_0^L dt(L-t)\int d^mK\ \Phi_\varepsilon(\mathbf{K})\cos(\mathbf{T}\cdot\mathbf{K})\times[(\eta K^2)^{-1}\sin(\eta K^2)-\cos(\eta K^2)]. \tag{3.18}$$

Further, for $L/kl_\varepsilon^2\gg 1$ Eq. (3.18) acquires a simplified form

$$\chi=k^2L\int_0^\infty dt\int d^mK\ \Phi_\varepsilon(\mathbf{K})\cos(\mathbf{T}\cdot\mathbf{K})\times[(tK^2/2k)^{-1}\sin(tK^2/2k)-\cos(tK^2/2k)]. \tag{3.19}$$

We may present this result as a convolution of the spectrum $\Phi_\varepsilon(\mathbf{K})$ with a filtering function $f(\mathbf{K})$:

$$\chi=\pi k^3L\int d^mK\ f(\mathbf{K})\Phi_\varepsilon(\mathbf{K}). \tag{3.20}$$

The filtering function is given by

$$f(\mathbf{K})=(\pi k)^{-1}\int_0^\infty dt\ \cos(tK\beta)[(tk^2/2k)^{-1}\sin(tK^2/2k)-\cos(tK^2/2k)], \tag{3.21}$$

where the parameter $\beta=|\cos(\mathbf{T},\mathbf{K})|$ is determined by the angle between the observation direction and the direction of the spatial vector $\mathbf{K}$. In order to perform the integration in Eq. (3.21) we introduce an auxiliary function

$$F_\nu(\mathbf{K})=(2/\pi)K^{-2}\int_0^\infty dt\ t^{-1}\cos(tK\beta)\sin(\nu tK^2/2k), \tag{3.22}$$

which allows us to rewrite the filtering function $f(\mathbf{K})$ as

$$f(\mathbf{K})=F_1(\mathbf{K})-[(\partial/\partial\nu)F_\nu(\mathbf{K})]_{\nu=1}. \tag{3.23}$$

Evaluation of the integral in Eq. (3.22) yields

$$F_\nu(\mathbf{K})=K^{-2}\vartheta(\nu K-2k\beta), \tag{3.24}$$

where $\vartheta(z)$ is the step function. Consequently, according to Eq. (3.23), $f(\mathbf{K})$ is expressed as

$$f(\mathbf{K})=K^{-2}\vartheta(K-2k\beta)-K^{-1}\delta(K-2k\beta). \tag{3.25}$$

In particular, for a one-dimensional system $\beta\equiv 1$ and the wave correction becomes

$$\chi=2\pi k^3L\left[\int_{2k}^\infty dK\ K^{-2}\Phi_\varepsilon(K)-(2k)^{-1}\Phi_\varepsilon(2k)\right]. \tag{3.26}$$

One can verify easily that in this case the correction is negative for any monotonically decreasing spectrum $\Phi_\varepsilon(K)$. The appearance of the specific spatial frequency $K=2k$ in the result is not surprising because it is simply the condition of the Bragg reflection resonance for a periodic lattice. Furthermore, we see that for spectra with a single characteristic scale l_ε, the effect is maximal for $k\sim 1/l_\varepsilon$, i.e., for wavelengths comparable with the correlation length of the medium. At low frequencies $\chi\sim\omega^2$ and the high-frequency behavior of the correction is governed by the $K\to\infty$ decrease of the spectrum $\Phi_\varepsilon(K)$. Being presented in terms of the localization length, which is inversely proportional to the correction obtained, this picture coincides exactly with the frequency dependence studied in [33] for the model of a randomly layered medium with continuously varying parameters. Restricting analysis of the 1D localization by these observations, we proceed with the study of two- and three-dimensional media, which are the main subject of the present work.

IV. ISOTROPIC MEDIA

For an isotropic spectrum $\Phi_\varepsilon(K)$ the wave correction in both two- and three-dimensional systems can be rewritten in the form

$$\chi=2\pi k^2L\int_0^\infty dK\ f(K)\Phi_\varepsilon(K), \tag{4.1}$$

where the scalar filtering function $f(K)$ depends on the dimensionality of the problem.

Two dimensions. In this case the filtering function is given by

$$f(K)=K\int_0^\infty dt\ J_0(tK)[(tK^2/2k)^{-1}\sin(tK^2/2k)-\cos(tK^2/2k)], \tag{4.2}$$

where $J_0(z)$ is the Bessel function. After integration we find

$$f(K)=\begin{cases}(2k/K)\arcsin(K/2k)-1/\sqrt{1-K^2/4k^2}, & K<2k\\ \pi k/K, & K\geqslant 2k.\end{cases} \tag{4.3}$$

For small values of the parameter $K/2k$ the filtering function has asymptotic behavior $f(K)=-K^2/12k^2$ and its absolute value increases with K, having a singularity at $K=2k$.

Three dimensions. In this case we get

$$f(K)=2K\int_0^\infty dt\ t^{-1}\sin(tK)[(tK^2/2k)^{-1}\sin(tK^2/2k)-\cos(tK^2/2k)]. \tag{4.4}$$

Evaluation of the integral leads to

$$f(K)=\begin{cases}0, & K<2k\\ 2\pi k, & K\geqslant 2k.\end{cases} \tag{4.5}$$

According to Eq. (4.5), only the spatial frequencies $K\geqslant 2k$, which are the sources of evanescent waves in the case of single scattering [3], give a contribution to the wave correction.

For a numerical example we use the simplest Gaussian correlation function

$$B_\varepsilon(R)=\sigma_\varepsilon^2\exp(-R^2/l_\varepsilon^2), \tag{4.6}$$

where σ_ε^2 is the dispersion of fluctuations and l_ε is the characteristic scale of inhomogeneities of the medium. Such a correlation function corresponds to the spectral density

$$\Phi_\varepsilon(K)=(2\sqrt{\pi})^{-m}l_\varepsilon^m\sigma_\varepsilon^2\exp(-l_\varepsilon^2K^2/4). \tag{4.7}$$

For this spectrum the wave correction in both two- and three-dimensional cases can be written as

$$\chi=s(\kappa)\ell\sigma_\varepsilon^2, \tag{4.8}$$

where we have introduced the dimensionless parameters

$$\kappa=kl_\varepsilon,\quad \ell=L/l_\varepsilon. \tag{4.9}$$

In two dimensions the normalized wave correction $s(\kappa)$ is defined as

$$s(\kappa)=(\pi/2)\kappa^3[E_1(\kappa)-\exp(-\kappa^2/2)I_0(\kappa^2/2)+F(\kappa)], \tag{4.10a}$$

where $E_1(z)$ is the exponential integral, $I_0(z)$ is the Bessel function, and $F(z)$ is given by

$$F(z)=(2/\pi)\int_0^1 dt\ t^{-1}\arcsin t\ \exp(-z^2t^2). \tag{4.10b}$$

In the three-dimensional case,

$$s(\kappa)=(\pi/2)\kappa^3\ \mathrm{erfc}(\kappa). \tag{4.11}$$

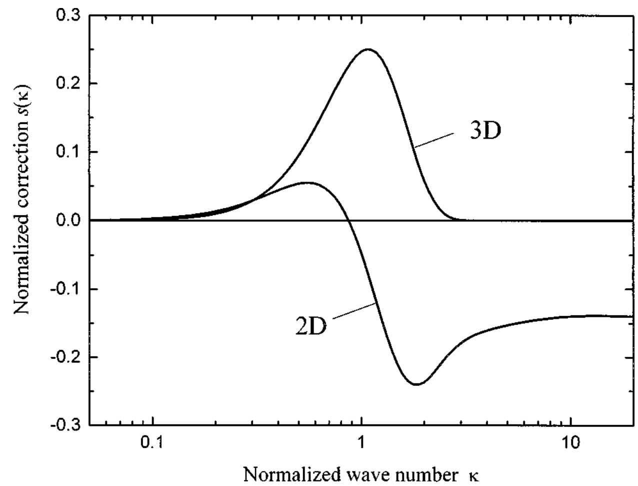

FIG. 1. Normalized wave correction in the 2D and 3D isotropic media as a function of the normalized wave number κ.

The dependence of the wave correction on the normalized wave number κ for both two- and three-dimensional problems is shown in Fig. 1. We see that there is an essential difference between two- and three-dimensional systems. For the 3D problem the wave correction in the far field is positive and has a quite narrow window in which the wavelength is comparable to the correlation scale l_ε. Roughly speaking, in this spectral window the medium behaves as a random resonant cavity accumulating the energy near the source. Within the framework of the wave localization concept, it is the intermediate spectral window that separates extended states at both higher and lower frequencies when the disorder is strong enough [13]. Irrespective of the actual possibility of strong localization in 3D systems, this result states the optimal conditions suitable for observing nonlinear effects or enhanced absorption in weakly dissipative random media.

In two dimensions there are negative values of the correction in the far field. This fact can serve as an additional independent indication of the possibility of strong Anderson-type localization in 2D random media. Unfortunately, our results, due to their asymptotic nature, cannot be a base for the exact prediction of localization; nevertheless, they could be a useful supplement to the results of other approaches, which, as a rule, only qualitatively describe the localization transition for classical waves. In particular, the advantage of our approach lies in its capability to trace the dependence of the correction on the correlation properties of the disorder, including such an interesting case as anisotropic media.

V. ANISOTROPIC MEDIA

In many situations the medium is characterized by an anisotropic spectrum of random inhomogeneities. The geophysical environment, such as Earth's subsurface, an underwater channel, or turbulent inhomogeneities in the atmosphere and ionosphere, with their predominantly layered structure, can serve as an example. All these media are characterized by inhomogeneous structures having the horizontal scales much greater than the vertical ones.

For the calculations in this case we use the general representation of the wave correction, Eq. (3.20), with filtering function (3.25). Further analytical simplification can be achieved for some given correlation functions only. The anisotropic generalization of the Gaussian correlation function (4.6) reads

$$B_\varepsilon(\mathbf{R})=\sigma_\varepsilon^2\exp[-z^2/l_\varepsilon^2-r^2/(\mu l_\varepsilon)^2],\quad \mu\geq 1, \tag{5.1}$$

where z denotes the vertical coordinate, $\mathbf{r}$ is the horizontal $(m-1)$-dimensional vector, and the parameter μ characterizes the degree of anisotropy of the medium. Particularly, for $\mu=1$ this correlation function reduces to its isotropic version (4.6). The correlation function (5.1) corresponds to the spectrum

$$\Phi_\varepsilon(\mathbf{K})=(2\sqrt{\pi})^{-m}\mu^{m-1}l_\varepsilon^m\sigma_\varepsilon^2\exp(-l_\varepsilon^2K_z^2/4-\mu^2l_\varepsilon^2K_r^2/4), \tag{5.2}$$

where K_z and $\mathbf{K}_r$ are, respectively, the vertical and horizontal components of the spatial wave number $\mathbf{K}$.

Two dimensions. In this case, introducing the polar coordinates gives

$$\chi=\pi k^3L\int_0^{2\pi}d\varphi\int_1^\infty dx[x^{-1}\Phi_\varepsilon(2k\beta x,\varphi)-\Phi_\varepsilon(2k\beta,\varphi)]. \tag{5.3}$$

Then, evaluating the integral over x, we obtain the same formula as Eq. (4.8), but with the coefficient $s(\kappa,\mu)$ depending on the anisotropy parameter μ as well,

$$s(\kappa,\mu)=\tfrac{1}{8}\mu\kappa^3\int_0^{2\pi}d\varphi[E_1(a^2\kappa^2)-2\exp(-a^2\kappa^2)]. \tag{5.4}$$

Here we have denoted

$$a^2\equiv a^2(\varphi,\varphi_0,\mu)=(\cos^2\varphi+\mu^2\sin^2\varphi)\cos^2(\varphi-\varphi_0) \tag{5.5}$$

and φ_0 is the observation angle. For isotropic media ($\mu=1$) the correction does not depend on φ_0 and Eq. (5.4) reduces to the relation

$$s(\kappa)=\tfrac{1}{2}\kappa^3\int_0^{\pi/2}d\varphi[E_1(\kappa^2\cos^2\varphi)-2\exp(-\kappa^2\cos^2\varphi)], \tag{5.6}$$

which represents another form of Eq. (4.10). It can be easily verified that even in the general case ($\mu\neq 1$) $ds/d\varphi_0=0$ for any value of φ_0. Thus the wave correction in a two-dimensional anisotropic medium does not depend on the observation angle, at least under the conditions of applicability of the asymptotic procedure used. The wave correction as a function of the normalized wave number κ for some values of the anisotropy parameter is plotted in Fig. 2. The effect of anisotropy causes only a change in the dependence on the wave number. It is interesting that for some range of μ the wave correction is greater than for an isotropic system.

Three dimensions. In this case, the wave correction in spherical coordinates has the form

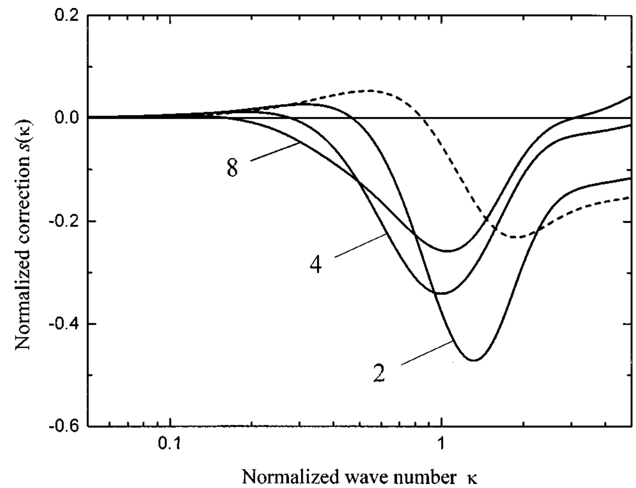

FIG. 2. Normalized wave correction in the 2D anisotropic medium for $\mu=2$, 4, and 8 as a function of the normalized wave number κ. The dashed line corresponds to the isotropic case ($\mu=1$).

$$\chi=2\pi k^4L\int_0^{2\pi}d\varphi\int_0^\pi d\theta\ \beta\ \sin\theta\int_1^\infty dx[\Phi_\varepsilon(2k\beta x,\varphi,\theta)-\Phi_\varepsilon(2k\beta,\varphi,\theta)]. \tag{5.7}$$

For the Gaussian spectrum (5.2) the correction term can be expressed by the same equation (4.8) with the coefficient s depending on the observation angle θ_0:

$$s(\kappa,\mu,\theta_0)=\tfrac{1}{8}\mu\kappa^3\int_0^{2\pi}d\varphi\int_0^\pi d\theta[1+(\mu\tan\theta)^{-2}]^{-1/2}\times[\mathrm{erfc}(a\kappa)-2\pi^{-1/2}a\kappa\exp(-a^2\kappa^2)], \tag{5.8}$$

where

$$a^2\equiv a^2(\varphi,\theta,\theta_0,\mu)=(\cos^2\theta+\mu^2\sin^2\theta)(\cos\theta\cos\theta_0+\sin\theta\sin\theta_0\cos\varphi)^2. \tag{5.9}$$

In three-dimensional media the directions along and across the quasilayered structure are essentially distinguished. The normalized wave correction as a function of the anisotropy parameter μ for the vertical direction (observation angle $\theta_0=0°$) is shown in Fig. 3(a). Above some value of μ the wave correction is negative and, consequently, for $\kappa\sim 1$ the mean intensity in the vertical direction is less than in the homogeneous medium. In Fig. 3(b) the same dependence is shown for the horizontal direction (observation angle $\theta_0=90°$). In this case we obtain the increase of the mean intensity for all degrees of anisotropy and for all wave-number values, which means the localization of the wave energy along the layers. This effect was investigated by many authors for purely layered random media [33–35]. Our results give additional information, permitting the investigation of the case intermediate between purely layered and isotropic systems.

In Fig. 4 the normalized wave correction is shown as a function of the observation angle. It is interesting that for a given value of μ there is a sector of observation angles for

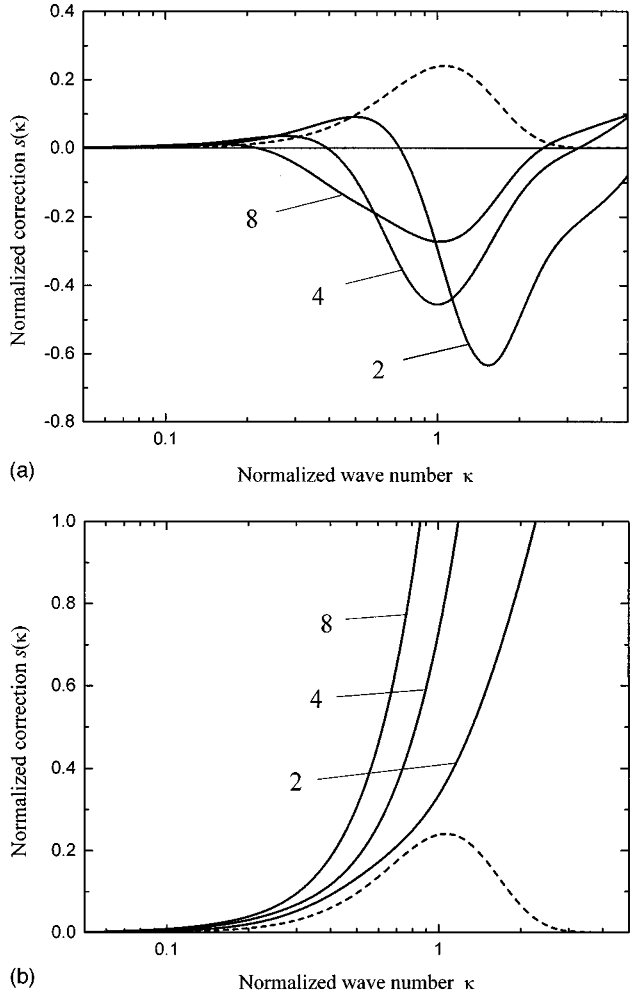

FIG. 3. Normalized wave correction in the 3D anisotropic medium for $\mu=2$, 4, and 8 as a function of the normalized wave number κ. The dashed line corresponds to the isotropic case ($\mu=1$). (a) Vertical direction (observation angle $\theta_0=0°$) and (b) horizontal direction (observation angle $\theta_0=90°$).

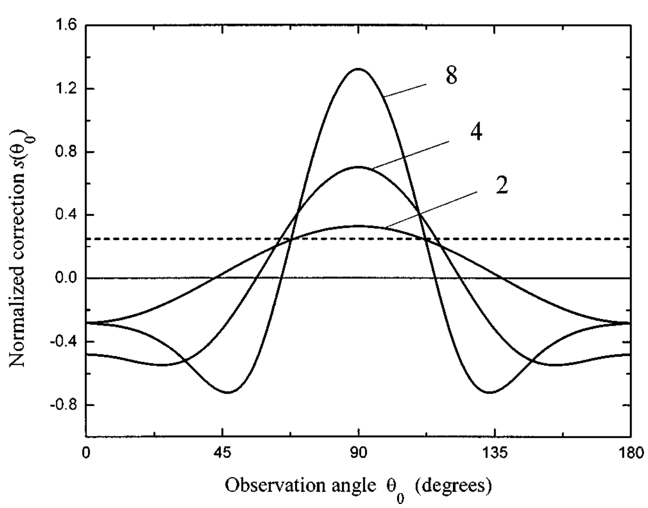

FIG. 4. Normalized wave correction in the 3D anisotropic medium for $\kappa=1$ and $\mu=2$, 4, and 8 as a function of the observation angle θ_0. The dashed line corresponds to the isotropic case ($\mu=1$).

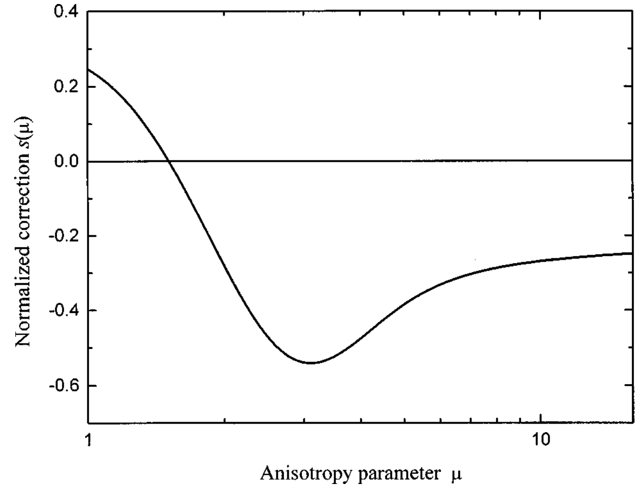

FIG. 5. Normalized wave correction in the 3D anisotropic medium for $\kappa=1$ and the vertical direction (observation angle $\theta_0=0°$) as a function of the anisotropy parameter μ.

which the field intensity decreases more quickly than in the vertical direction. In Fig. 5 we present the wave correction as a function of the anisotropy parameter for the vertical direction. This dependence shows that there is a critical value of μ (which is equal to about 1.7), below which the medium behaves basically as a three-dimensional isotropic medium, i.e., the wave correction is positive for all observation angles. Above this critical value the properties of the medium are similar to those of a layered structure. Another interesting feature is the existence of a finite value of μ (for the Gaussian spectrum this value is equal to 3.1) for which the localization effect is maximal in some sense.

VI. SUMMARY AND CONCLUSIONS

In this paper we have implemented the Feynman path-integral approach to the problem of classical wave propagation in random media. For this purpose we have applied the method originally proposed by Fock for the integration of quantum-mechanical equations. The method is based on the introduction of an additional pseudotime variable and the transfer to a higher-dimensional space in which the propagation process is described by a generalized parabolic equation similar to the nonstationary Schrödinger equation in quantum mechanics. The advantage of such a transfer, which can be interpreted also as a version of the embedding technique, is that it allows us to present the solution of the parabolic-type equation in a functional-integral form.

Applying the generalized path-integral solution, we have examined the mean intensity of the field excited by a point source in a statistically homogeneous Gaussian random medium. By using a perturbative technique we have shown that the normalized mean intensity in the far field can be presented as the sum of the leading term and some correction, which is related physically to the coherent backscattering and repeated multiple scattering by the same inhomogeneities. In diagrammatic language, this means that in addition to ladder diagrams, our perturbative approach accounts, even in the first-order approximation, for all possible diagrams, including maximally crossed and ‘‘recurrent’’ones, which are responsible for the localization mechanisms. In fact, as shown, the dependence of the correction term on the wave number has a quite narrow peak centered at the typical spatial frequency in the random medium spectrum. This dependence does not differ significantly from that obtained in classical works concerning wave localization in discrete random media.

The results indicate that there is an essential difference between two- and three-dimensional systems. While in three dimensions the wave correction in the far field is positive, for 2D systems there are negative values of the correction. This fact can serve as an additional independent confirmation of the possibility of strong Anderson-type localization in 2D random media. In the case of anisotropic spectra the behavior of the correction for 2D and 3D systems is also different. In two dimensions the wave correction does not depend on the observation angle. In three-dimensional media the directions along and across the quasilayered structure are essentially distinguished. Moreover, the investigation of the wave correction as a function of anisotropy parameter shows that there is a critical value of μ, below which the medium behaves basically as a three-dimensional isotropic medium, i.e., the wave correction is positive for all observation angles. Above this critical value the localization of the wave energy along the quasilayered structure is observed.

Hence the technique applied in our work allowed us to account for coherent backscattering effects and, by evaluating the wave correction for the mean intensity of the field, to study the localization properties of randomly perturbed continuous media. We also hope that the limitations of the approach, due to the asymptotic procedures used, could be mitigated by application of direct numerical techniques, particularly Monte Carlo methods, to the evaluation of the functional integrals.

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